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On Semi-Invariant Submanifolds of a Nearly Hyperbolic Kenmotsu Manifold with Semi-symmetric Metric Connection

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Abstract
We consider a nearly hyperbolic Kenmotsu manifold admitting a semi-symmetric metric connection and study semi-invariant submanifolds of a nearly hyperbolic Kenmotsu manifold with semi-symmetric metric connection. We also find the integrability conditions of some distributions on nearly hyperbolic Kenmotsu manifold andstudy parallel distributions on nearly hyperbolic Kenmotsu manifold.
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I. Introduction

Let ∇ be a linear connection in an n-dimensional differentiable manifold $\bar{M}$. The torsion tensor T and curvature tensor R of ∇ given respectively by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The connection ∇ is symmetric if its torsion tensor T vanishes, otherwise it is non-symmetric. The connection ∇ is metric connection if there is a Riemannian metric g in $\bar{M}$ such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection. A. Friedmann and J. A. Schouten introduced the idea of a semi-symmetric connection. A linear connection is said to be a semi-symmetric connection if its torsion tensor T is of the form:

$$T(X,Y) = \eta(X)Y - \eta(Y)X.$$

Many geometers (see,[20], [21]) have studied some properties of semi-symmetric metric connection.

The study of CR-submanifolds of Kaehler manifold as generalization of invariant and anti-invariant submanifolds was initiated by A. Bejancu in [8]. A semi-invariant submanifold is the extension of a CR-submanifold of a Kaehler manifold to submanifolds of almost contact manifolds. The study of semi-invariant submanifolds of Sasakian manifolds was initiated by Bejancu-Papaghuic in [10]. The same concept was studied under the name contact CR-submanifold by Yano-Kon in [19] and K. Matsumoto in [16].The study of semi-invariant submanifolds in almost contact manifold was enriched by several geometers (see, [2], [4], [5], [6], [7], [8], [12], [13], [17]). On the otherhand, almost hyperbolic (f,ξ,η,g) -structure was defined and studied by Upadhyay and Dube in [18].Joshi and Dube studied semi-invariant submanifolds of an almost r-contact hyperbolic metric manifold in [15]. Ahmad M., K. Ali, studied semi-invariant submanifolds of a nearly hyperbolic Kenmotsu manifold with a semi-symmetric non-metric connection in [2]. In this paper, we study semi-invariant submanifolds of a nearly hyperbolic Kenmotsu manifold admitting a semi-symmetric metric connection.

This paper is organized as follows. In section 2, we give a brief description of nearly hyperbolic Kenmotsu manifold admitting a semi-symmetric metric connection. In section 3, we study some properties of semi-invariant submanifolds of a nearly hyperbolic Kenmotsu manifold with a semi-symmetric metric connection. In section 4,we discuss the integrability conditions of some distributions on nearly hyperbolic Kenmotsu manifold with a semi-symmetric metric connection. In section 5, we study parallel horizontal distribution on nearly hyperbolic Kenmotsu manifold with a semi-symmetric metric connection.

II. Preliminaries

Let $\bar{M}$ be an n-dimensional almost hyperbolic contact metric manifold with the almost hyperbolic contact metric structure$(\emptyset, \xi, \eta, g)$, where a tensor $\emptyset$ of type (1,1), a vector field ξ, called structure vector field and η, the dual 1-form of $\xi$and the associated Riemannian metric $g$satisfying the following

$$\emptyset^2 X = X + \eta(X)\xi, \qquad g(X, \xi) = \eta(X) \tag{2.1}$$

$$\eta(\xi) = -1, \qquad \emptyset(\xi) = 0, \qquad \eta o \emptyset = 0, \tag{2.2}$$

$$g(\emptyset X, \emptyset Y) = -g(X, Y) - \eta(X)\eta(Y) \tag{2.3}$$

for any X, Y tangent to $\bar{M}$ [19]. In this case

$$g(\emptyset X, Y) = -g(\emptyset Y, X). \tag{2.4}$$

An almost hyperbolic contact metric structure $(\emptyset, \xi, \eta, g)$ on $\bar{M}$ is called hyperbolic Kenmotsu manifold [7] if and only if

$$(\nabla_X \emptyset)Y = g(\emptyset X, Y) - \eta(Y)\emptyset X \tag{2.5}$$

for all X, Y tangent to $\bar{M}$.

On a hyperbolic Kenmotsu manifold$\bar{M}$, we have

$$\nabla_X \xi = X + \eta(X)\xi \tag{2.6}$$

for a Riemannian metric g and Riemannian connection∇.

Further, an almost hyperbolic contact metric manifold $\bar{M}$ on $(\emptyset, \xi, \eta, g)$ is called a nearly hyperbolic Kenmotsu manifold [7], if

$$(\nabla_X \emptyset)Y + (\nabla_Y \emptyset)X = -\eta(X)\emptyset Y - \eta(Y)\emptyset X, \tag{2.7}$$

where∇ is Riemannian connection $\bar{M}$.

Now, we define a semi-symmetric metric connection

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi \tag{2.8}$$

Such that$(\bar{\nabla}_X g)(Y, Z) = 0$.From(2.7) and(2.8), replacing $Y$by $\emptyset Y$, we have

$$(\bar{\nabla}_X \emptyset)Y + (\bar{\nabla}_Y \emptyset)X = -2\eta(X)\emptyset Y - 2\eta(Y)\emptyset X \tag{2.9}$$

$$\bar{\nabla}_X \xi = 0. \tag{2.10}$$

An almost hyperbolic contact metric manifold with almost hyperbolic contact structure $(\emptyset, \xi, \eta, g)$ is called nearly hyperbolic Kenmotsu manifold with semi-symmetric metric connection if it is satisfied (2.9) and (2.10).

Let Riemannian metric symbol g and ∇ be induced Levi-Civita connection on M then the Guass formula and Weingarten formula is given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.11}$$

$$\bar{\nabla}_X N = -A_N X + \nabla_X^{\perp} N \tag{2.12}$$

for any $X, Y \in TM$ and $N \in T^{\perp}M$, where $\nabla^{\perp}$ is a connection on the normal bundle $T^{\perp}M$, $h$is the second fundamental form and A_N is the Weingarten map associated with N as

$$g(h(X,Y),N) = g(A_N X, Y). \quad (2.13)$$

Any vector X tangent to M is given as

$$X = PX + QX + \eta(X)\xi, \quad (2.14)$$

where$PX \in D \; and \; QX \in D^{\perp}$.

For any N normal to M, we have

$$\emptyset N = BN + CN, \quad (2.15)$$

whereBN $(resp. CN)$ is the tangential component $(resp. normal\ component)$ of $\emptyset N$.

For almost contact structure the Nijenhuis tensor $N(X,Y)$ is expressed as

$$N(X,Y) = (\bar{\nabla}_{\emptyset X}\emptyset)Y - (\bar{\nabla}_{\emptyset Y}\emptyset)X - \emptyset(\bar{\nabla}_X\emptyset)Y + \emptyset(\bar{\nabla}_Y\emptyset)X \quad (2.16)$$

Now from (2.9),replacing X by $\emptyset X$, we have

$$(\overline{\nabla}_{\emptyset X}\emptyset)Y = -\eta(Y)X - \eta(X)\eta(Y)\xi - (\overline{\nabla}_Y\emptyset)\emptyset X. \quad (2.17)$$

Differentiating (2.1) conveniently along the vector and using (2.10), we have

$$(\overline{\nabla}_Y\emptyset)\emptyset X = (\overline{\nabla}_Y\eta)(X)\xi - \emptyset(\overline{\nabla}_Y\emptyset)X. \quad (2.18)$$

From (2.17) and (2.18), we have

$$(\overline{\nabla}_{\emptyset X}\emptyset)Y = -\eta(Y)X - \eta(X)\eta(Y)\xi - (\overline{\nabla}_Y\eta)(X)\xi + \emptyset(\overline{\nabla}_Y\emptyset)X. \quad (2.19)$$

Interchanging $X$and Y, we have

$$(\overline{\nabla}_{\emptyset Y}\emptyset)X = -\eta(X)Y - \eta(X)\eta(Y)\xi - (\overline{\nabla}_X\eta)(Y)\xi + \emptyset(\overline{\nabla}_X\emptyset)Y. \quad (2.20)$$

Using equation (2.19), (2.20) and (2.9) in (2.16), we have

$$N(X,Y) = \eta(X)Y - \eta(Y)X + 2d\eta(X,Y)\xi + 2\emptyset(\overline{\nabla}_Y\emptyset)X - 2\emptyset(\overline{\nabla}_X\emptyset)Y, \quad (2.21)$$

$$N(X,Y) = 2g(\emptyset X,Y)\xi + 5\eta(X)Y + 3\eta(Y)X + 8\eta(X)\eta(Y)\xi + 4\emptyset(\overline{\nabla}_Y\emptyset)X. \quad (2.22)$$

As we know that,$(\overline{\nabla}_Y\emptyset)X = \overline{\nabla}_Y\emptyset X - \emptyset(\overline{\nabla}_Y X)$.UsingGuass formula (2.11), we have

$$\emptyset(\overline{\nabla}_Y\emptyset)X = \emptyset(\nabla_Y\emptyset X) + \emptyset h(Y,\emptyset X) - \nabla_Y X - \eta(\nabla_Y X)\xi - h(Y,X).$$

Using this equation in (2.22), we have

$$N(X,Y) = 2g(\emptyset X,Y)\xi + 5\eta(X)Y + 3\eta(Y)X + 8\eta(X)\eta(Y)\xi + 4\emptyset(\nabla_Y\emptyset X)$$
$$+4\emptyset h(Y,\emptyset X) - 4(\nabla_Y X) - 4\eta(\nabla_Y X)\xi - 4h(Y,X). \quad (2.23)$$

III. Semi-invariant Submanifold

Let M be submanifold immersed in $\overline{M}$, we assume that the vector ξ is tangent to M, denoted by $\{\xi\}$ the 1-dimentional distribution spanned by ξ on M, then M is called a semi-invariant submanifold [9] of M if there exist two differentiable distribution D &$D^{\perp}$ on M satisfying

(i) $TM = D \oplus D^{\perp} \oplus \xi$, where D , $D^{\perp}$& ξ are mutually orthogonal to each other,

(ii) the distribution$D$ is invariant under $\emptyset$that is $\emptyset Dx = Dx$ for all $x \in M$,

(iii) the distribution$D^{\perp}$is anti-invariant under $\emptyset$, that is $\emptyset D^{\perp}x \subset T^{\perp}M$ for all$x \in M$,where TM & $T^{\perp}M$ be the Lie algebra of vector fields tangential & normal to M respectively.

Theorem 3.1. The connection induced on semi-invariant submanifold of nearly hyperbolic Kenmotsu manifold with semi-symmetric metric connection also semi-symmetric metric.

Proof:Let ∇be induced connection with respect to the normal N on semi-invariant submanifold of nearly hyperbolic Kenmotsu manifold with semi-symmetric metricconnection $\overline{\nabla}$, then

$$\overline{\nabla}_X Y = \nabla_X Y + m(X,Y), \tag{3.1}$$

where m is the tensor filed of type (0,2) on semi-invariant submanifold M. If ∇^* be the induced connection on semi-invariant submanifold from Riemannian connection $\overline{\overline{\nabla}}$,then

$$\overline{\overline{\nabla}}_X Y = \nabla^*_X Y + h(X,Y), \tag{3.2}$$

where$h$ is second fundamental tensor and we know that semi-symmetric metric connection on nearly hyperbolic Kenmotsu manifold

$$\overline{\nabla}_X Y = \overline{\overline{\nabla}}_X Y + (Y)X - g(X,Y)\xi. \tag{3.3}$$

Using (3.1), (3.2) in (3.3), we have

$$\nabla_X Y + m(X,Y) = \nabla^*_X Y + h(X,Y) + \eta(Y)X - g(X,Y)\xi. \tag{3.4}$$

Comparing tangent and normal parts, we have

$$\nabla_X Y = \nabla^*_X Y + \eta(Y)X - g(X,Y)\xi,$$
$$m(X,Y) = h(X,Y).$$

Thus ∇ is also semi-symmetric metric connection.

Lemma 3.2. Let M be a semi-invariant submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with semi-symmetric metric connection, then

$$2(\overline{\nabla}_X \emptyset)Y = \nabla_X \emptyset Y - \nabla_Y \emptyset X + h(X,\emptyset Y) - h(Y,\emptyset X) - \emptyset[X,Y]$$

for each $X, Y \in D$.

Proof.By Gauss formula (2.11), we have

$$\overline{\nabla}_X \emptyset Y - \overline{\nabla}_Y \emptyset X = \nabla_X \emptyset Y - \nabla_Y \emptyset X + h(X,\emptyset Y) - h(Y,\emptyset X). \tag{3.5}$$

Also, by covariantly differentiation, we know that

$$\overline{\nabla}_X \emptyset Y - \overline{\nabla}_Y \emptyset X = (\overline{\nabla}_X \emptyset)Y - (\overline{\nabla}_Y \emptyset)X + \emptyset[X,Y]. \tag{3.6}$$

From (3.5) and (3.6), we have

$$(\overline{\nabla}_X \emptyset)Y - (\overline{\nabla}_Y \emptyset)X = \nabla_X \emptyset Y - \nabla_Y \emptyset X + h(X,\emptyset Y) - h(Y,\emptyset X) - \emptyset[X,Y]. \tag{3.7}$$

Adding (2.9) and (3.7), we obtain

$$2(\overline{\nabla}_X \emptyset)Y = \nabla_X \emptyset Y - \nabla_Y \emptyset X + h(X,\emptyset Y) - h(Y,\emptyset X) - \emptyset[X,Y]$$

for each $X, Y \in D$.

Lemma 3.3. Let M be a semi-invariant submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with semi-symmetric metric connection, then

$$2(\overline{\nabla}_Y \emptyset)X = \nabla_Y \emptyset X - \nabla_X \emptyset Y + h(Y,\emptyset X) - h(X,\emptyset Y) + \emptyset[X,Y]$$

for each $X, Y \in D$.

Theorem 3.4. Let M be a semi-invariant submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with semi-symmetric metric connection, then

$$2(\overline{\nabla}_X \emptyset)Y = -A_{\emptyset Y}X + \nabla^{\perp}_X \emptyset Y - \nabla_Y \emptyset X - h(Y,\emptyset X) - \emptyset[X,Y]$$

for all X $\in D$ and $Y \in D^{\perp}$.

Proof.By Gauss formulas (2.11), we have

$$\overline{\nabla}_Y \emptyset X = \nabla_Y \emptyset X + h(Y,\emptyset X).$$

Also, by Weingarten formula (2.12), we have

$$\overline{\nabla}_X \emptyset Y = -A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y.$$

From abovetwo equations, we have

$$\overline{\nabla}_X \emptyset Y - \overline{\nabla}_Y \emptyset X = -A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \nabla_Y \emptyset X - h(Y, \emptyset X). \quad (3.8)$$

Comparing equation (3.6) and (3.8), we have

$$(\overline{\nabla}_X \emptyset)Y - (\overline{\nabla}_Y \emptyset)X = -A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \nabla_Y \emptyset X - h(Y, \emptyset X) - \emptyset[X, Y]. \quad (3.9)$$

Adding equation (2.9) in (3.9), we have

$$2(\overline{\nabla}_X \emptyset)Y = -A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \eta(X)Y - \nabla_Y \emptyset X - h(Y, \emptyset X) - \emptyset[X, Y] - 2\eta(X)\emptyset Y - 2\eta(Y)\emptyset X$$

$$2(\overline{\nabla}_X \emptyset)Y = -A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \nabla_Y \emptyset X - h(Y, \emptyset X) - \emptyset[X, Y]$$

for all $X \in D$ and $Y \in D^{\perp}$.

Lemma 3.5. Let M be a semi-invariant submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with semi-symmetric metric connection, then

$$2(\overline{\nabla}_Y \emptyset)X = A_{\emptyset Y}X - \nabla_X^{\perp}\emptyset Y + \nabla_Y \emptyset X + h(Y, \emptyset X) + \emptyset[X, Y] \quad (3.10)$$

for all$X \in D$ and $Y \in D^{\perp}$.

Lemma 3.6. Let M be a semi-invariant submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with semi-symmetric metric connection, then

$$2(\overline{\nabla}_X \emptyset)Y = A_{\emptyset X}Y - A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \nabla_Y^{\perp}\emptyset X - \emptyset[X, Y] \quad (3.11)$$

for all $X, Y \in D^{\perp}$.

Proof.Using Weingarten formula (2.12), we have

$$\overline{\nabla}_X \emptyset Y - \overline{\nabla}_Y \emptyset X = A_{\emptyset X}Y - A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \nabla_Y^{\perp}\emptyset X.$$

Using equation (3.6) in above, we have

$$(\overline{\nabla}_X \emptyset)Y - (\overline{\nabla}_Y \emptyset)X = A_{\emptyset X}Y - A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \nabla_Y^{\perp}\emptyset X - \emptyset[X, Y]. \quad (3.12)$$

Adding (2.9) in above, we have

$$2(\overline{\nabla}_X \emptyset)Y = A_{\emptyset X}Y - A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \nabla_Y^{\perp}\emptyset X - \emptyset[X, Y]$$

for all $X, Y \in D^{\perp}$.

Lemma 3.7. Let M be a semi-invariant submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with semi-symmetric metric connection, then

$$2(\overline{\nabla}_Y \emptyset)X = A_{\emptyset Y}X - A_{\emptyset X}Y + \nabla_Y^{\perp}\emptyset X - \nabla_X^{\perp}\emptyset Y + \emptyset[X, Y]$$

for all$X, Y \in D^{\perp}$

Lemma 3.8. Let M be a semi-invariant submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with semi-symmetric metricconnection.Then

$$P(\nabla_X \emptyset PY) + P(\nabla_Y \emptyset PX) - PA_{\emptyset QY}X - PA_{\emptyset QX}Y = \emptyset P(\nabla_X Y) + \emptyset P(\nabla_Y X)$$
$$-2\eta(X)\emptyset PY - 2\eta(Y)\emptyset PX, \quad (3.13)$$

$$Q(\nabla_X \emptyset PY) + Q(\nabla_Y \emptyset PX) - QA_{\emptyset QY}X - QA_{\emptyset QX}Y = 2Bh(X, Y), \quad (3.14)$$

$$h(Y, \emptyset PX) + h(X, \emptyset PY) + \nabla_X^{\perp}\emptyset QY + \nabla_Y^{\perp}\emptyset QX = \emptyset Q(\nabla_X Y) + \emptyset Q(\nabla_Y X)$$
$$+2Ch(X, Y) - 2\eta(X)\emptyset QY - 2\eta(Y)\emptyset QX, \quad (3.15)$$

$$\eta(\nabla_X \emptyset PY) + \eta(\nabla_Y \emptyset PX) - \eta(A_{\emptyset QY}X) - \eta(A_{\emptyset QX}Y) = 0 \quad (3.16)$$

for all $X, Y \in TM$.

Proof.Differentiating covariantlyequation (2.14) and using equation (2.11) and(2.12), we have

$$(\overline{\nabla}_X \emptyset)Y + \emptyset(\nabla_X Y) + \emptyset h(X,Y) = \nabla_X \emptyset PY + h(X,\emptyset PY) - A_{\emptyset QY}X + \nabla_X^{\perp} \emptyset QY.$$

Interchanging $X$and Y, we have

$$(\overline{\nabla}_Y \emptyset)X + \emptyset(\nabla_Y X) + \emptyset h(Y,X) = \nabla_Y \emptyset PX + h(Y,\emptyset PX) - A_{\emptyset QX}Y + \nabla_Y^{\perp} \emptyset QX.$$

Adding above two equations, we have

$$(\overline{\nabla}_X \emptyset)Y + (\overline{\nabla}_Y \emptyset)X + \emptyset(\nabla_X Y) + \emptyset(\nabla_Y X) + 2\emptyset h(X,Y) = \nabla_X \emptyset PY + \nabla_Y \emptyset PX$$
$$+h(X,\emptyset PY) + h(Y,\emptyset PX) - A_{\emptyset QY}X - A_{\emptyset QX}Y + \nabla_X^{\perp} \emptyset QY + \nabla_Y^{\perp} \emptyset QX.$$

Using equation (2.9) in above, we have

$$-2\eta(X)\emptyset Y - 2\eta(Y)\emptyset X + \emptyset(\nabla_X Y) + \emptyset(\nabla_Y X) + 2\emptyset h(X,Y) = \nabla_X \emptyset PY + \nabla_Y \emptyset PX$$
$$+h(X,\emptyset PY) + h(Y,\emptyset PX) - A_{\emptyset QY}X - A_{\emptyset QX}Y + \nabla_X^{\perp} \emptyset QY + \nabla_Y^{\perp} \emptyset QX.$$

Using equations (2.14), (2.15)&(2.2), we have

$$-2\eta(X)\emptyset PY - 2\eta(X)\emptyset QY - 2\eta(Y)\emptyset PX - 2\eta(Y)\emptyset QX + \emptyset P(\nabla_X Y) + \emptyset Q(\nabla_X Y)$$
$$+\emptyset P(\nabla_Y X) + \emptyset Q(\nabla_Y X) + 2Bh(X,Y) + 2Ch(X,Y) = P(\nabla_X \emptyset PY) + Q(\nabla_X \emptyset PY)$$
$$+\eta(\nabla_X \emptyset PY)\xi + P(\nabla_Y \emptyset PX) + Q(\nabla_Y \emptyset PX) + \eta(\nabla_Y \emptyset PX)\xi + h(X,\emptyset PY)$$
$$+h(Y,\emptyset PX) - PA_{\emptyset QY}X - QA_{\emptyset QY}X - \eta\left(A_{\emptyset QY}X\right)\xi - PA_{\emptyset QX}Y - QA_{\emptyset QX}Y$$
$$-\eta\left(A_{\emptyset QX}Y\right)\xi + \nabla_X^{\perp} \emptyset QY + \nabla_Y^{\perp} \emptyset QX.$$

Comparing horizontal, vertical and normal components we get desired results.

⊡

IV. Integrability of Distributions

Theorem4.1. Let M be a semi-invariant submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with semi-symmetric metric connection, then the distribution $D\oplus\langle\xi\rangle$ is integrable if the following conditions are satisfied

$$S(X,Y) \in (D\oplus\langle\xi\rangle) \tag{4.1}$$

$$h(X,\emptyset Y) = h(\emptyset X,Y) \tag{4.2}$$

for each $X,Y \in (D\oplus\langle\xi\rangle)$.

Proof.The torsion tensor $S(X,Y)$ of an almost hyperbolic contact manifold is given by

$$S(X,Y) = N(X,Y) + 2d\eta(X,Y)\xi,$$

where $N(X,Y)$ is Neijenhuis tensor,

$$S(X,Y) = [\emptyset X,\emptyset Y] - \emptyset[\emptyset X,Y] - \emptyset[X,\emptyset Y] + 2d\eta(X,Y)\xi. \tag{4.3}$$

Suppose that $(D\oplus\langle\xi\rangle)$ is integrable, then$N(X,Y) = 0$for any $X,Y \in (D\oplus\langle\xi\rangle)$.

Therefore, $S(X,Y) = 2d\eta(X,Y)\xi \in (D\oplus\langle\xi\rangle)$.

From (4.3), (2.23) and comparing normal part, we have

$$\emptyset Q(\nabla_Y \emptyset X) + Ch(Y,\emptyset X) - h(X,Y) = 0$$

for $X,Y \in (D\oplus\langle\xi\rangle)$.

Replacing$Y$ by $\emptyset Z$, where $Z \in D$,

$$\emptyset Q(\nabla_{\emptyset Z} \emptyset X) + Ch(\emptyset Z,\emptyset X) - h(X,\emptyset Z) = 0. \tag{4.4}$$

Interchanging X and Z, we have

$$\emptyset Q(\nabla_{\emptyset X} \emptyset Z) + Ch(\emptyset X,\emptyset Z) - h(Z,\emptyset X) = 0. \tag{4.5}$$

Subtracting (4.4) from (4.5), we obtain

$$\emptyset Q[\emptyset X, \emptyset Z] + h(X, \emptyset Z) - h(Z, \emptyset X) = 0.$$

Since $D \oplus \langle \xi \rangle$ is integrable so that $[\emptyset X, \emptyset Z] \in (D \oplus \langle \xi \rangle)$for $X, Y \in D$.

Consequently, above equation gives

$$h(X, \emptyset Z) = h(\emptyset X, Z).$$

⊡

Proposition 4.2. Let M be a semi-invariant submanifold of a nearly hyperbolicKenmotsu manifold $\bar{M}$ with semi-symmetric metricconnection, then

$$A_{\emptyset Y} Z - A_{\emptyset Z} Y = \frac{1}{3} \emptyset P[Y, Z]$$

for each $Y, Z \in D^{\perp}$.

Proof.Let $Y, Z \in D^{\perp}$and $X \in TM$, from (2.13), we have

$$2g(A_{\emptyset Z} Y, X) = g(h(Y, X), \emptyset Z) + g(h(X, Y), \emptyset Z). \quad (4.6)$$

Using (2.11) and (2.9) in (4.6), we have

$$2\, g(A_{\emptyset Z} Y, X) = -g(\nabla_X \emptyset Y, Z) - g(\nabla_Y \emptyset X, Z) - g(h(X, \emptyset Y) + h(Y, \emptyset X), Z)$$
$$-2\eta(X) g(\emptyset Y, Z) - 2\eta(Y) g(\emptyset X, Z). \quad (4.7)$$

From (2.12) and (4.7), we have

$$2g(A_{\emptyset Z} Y, X) = -g(\emptyset \nabla_Y Z, X) + g(A_{\emptyset Y} Z, X).$$

Transvecting$X$ from both sides, we obtain

$$2A_{\emptyset Z} Y = -\emptyset \nabla_Y Z + A_{\emptyset Y} Z. \quad (4.8)$$

Interchanging$Y$ and Z, we have

$$2A_{\emptyset Y} Z = -\emptyset \nabla_Z Y + A_{\emptyset Z} Y. \quad (4.9)$$

Subtracting(4.8) from (4.9), we have

$$(A_{\emptyset Y} Z - A_{\emptyset Z} Y) = \frac{1}{3} \emptyset P[Y, Z]. \quad (4.10)$$

Theorem4.3. Let M be a semi-invariant submanifold of a nearly hyperbolic Kenmotsu manifold $\bar{M}$ with semi-symmetric metric connection, then the distribution is integrable if and only if

$$A_{\emptyset Y} Z - A_{\emptyset Z} Y = 0 \quad (4.11)$$

for all $Y, Z \in D^{\perp}$.

Proof.Suppose that the distribution $D^{\perp}$ is integrable, that is $[Y, Z] \in D^{\perp}$. Therefore, $P[Y, Z] = 0$ for any $Y, Z \in D^{\perp}$.

Consequently, from (4.10) we have

$$A_{\emptyset Y} Z - A_{\emptyset Z} Y = 0.$$

Conversely, let (4.11) holds. Then by virtue of (4.10), we have either $P[Y, Z] = 0$ or $P[Y, Z] = k\xi$.But$P[Y, Z] = k\xi$is not possible as P being a projection operator on D.So,$P[Y, Z] = 0$.This implies that$[Y, Z] \in D^{\perp}$for all $Y, Z \in D^{\perp}$.

Hence $D^{\perp}$ is integrable.

V. Parallel Distribution

Definition 5.1. The horizontal (resp., vertical) distribution $D(resp., D^{\perp})$ is said to be parallel [7] with respect to the connection on $M\ if\ \nabla_X Y \in D\ (resp., \nabla_Z W \in D^{\perp})$ for any vector field.

Proposition 5.2. Let M be a semi-invariant submanifold of a nearly hyperbolicKenmotsu manifold $\bar{M}$ with semi-symmetric metric connection. If the horizontal distribution D is parallel then$h(X, \emptyset Y) = h(Y, \emptyset X)$for all$X, Y \in D$.

Proof.Let$X, Y \in D$, as$D$ is parallel distribution.So that $\nabla_X \emptyset Y \in Dand\nabla_Y \emptyset X \in D$.

Now, from (3.14) and (3.15), we have

$$Q(\nabla_X \emptyset PY) + Q(\nabla_Y \emptyset PX) - QA_{\emptyset QY}X - QA_{\emptyset QX}Y + h(Y, \emptyset PX) + h(X, \emptyset PY)$$
$$+\nabla_X^{\perp} \emptyset QY + \nabla_Y^{\perp} \emptyset QX = 2Bh(X, Y) + \emptyset Q(\nabla_X Y) + \emptyset Q(\nabla_Y X) + 2Ch(X, Y)$$
$$-2\eta(X)\emptyset QY - 2\eta(Y)\emptyset QX.$$

AsQ being a projection operator on $D^{\perp}$, then we have

$$h(X, \emptyset PY) + h(Y, \emptyset PX) = 2Bh(X, Y) + 2Ch(X, Y).$$

Using (2.14) and (2.15) in above, we have

$$h(X, \emptyset Y) + h(Y, \emptyset X) = 2\emptyset h(X, Y). \tag{5.1}$$

Replacing$X\ by\ \emptyset X$ in (5.1) and using (2.1), we have

$$h(\emptyset X, \emptyset Y) + h(Y, X) = 2\emptyset h(\emptyset X, Y). \tag{5.2}$$

Replacing$Y\ by\ \emptyset Y$ in (5.1) and using (2.1), we have

$$h(X, Y) + h(\emptyset Y, \emptyset X) = 2\emptyset h(X, \emptyset Y). \tag{5.3}$$

Comparing (5.2) and (5.3), we get

$$h(X, \emptyset Y) = h(Y, \emptyset X)$$

for all$X, Y \in D$.

Definition 5.3.A Semi-invariant submanifold is said to be mixed totally geodesic if $h(X, Y) = 0$ for all$X \in D$ and $Y \in D^{\perp}$.

Proposition 5.4. Let M be a semi-invariant submanifold of a nearly hyperbolicKenmotsu manifold $\bar{M}$ with semi-symmetric metric connection. Then M is a mixed totally geodesic if and only if $A_N X \in D\ for\ all\ X \in D$.

Proof.Let $A_N X \in D$ for all$X \in D$.

Now, $g(h(X, Y), N) = g(A_N X, Y) = 0\ \ for\ Y \in D^{\perp}$, which is equivalent to $h(X, Y) = 0$. Hence M is totally mixed geodesic.

Conversely, let M is totally mixed geodesic.That is $h(X, Y) = 0\ for\ X \in D$ and $Y \in D^{\perp}$.

Now, from$g(h(X, Y), N) = g(A_N X, Y)$, we get $g(A_N X, Y) = 0$. That is$A_N X \in D$ for all $Y \in D^{\perp}$.

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